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The quantum mechanical Schrödinger picture of a q-oscillator

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Abstract. A q-deformed version of standard quantum mechanics in the coordinate Schrödinger picture is obtained by replacing the ordinary coordinate derivative by the so-called q-discrete derivative as the representative of the momentum operator. The chosen q-discrete derivative is symmetric with respect to the exchange of q and q^{-1}. Under the usually adopted assumptions a q-deformed Schrödinger equation is derived for a harmonic oscillator. The complete set of eigenfunctions can be explicitly constructed as special q-functions and the corresponding energy eigenvalues are identical to those obtained by Biedenharn in his pioneering work. This q-deformed oscillator exhibits a rich novel structure including dynamical symmetry and in the limit $q \to 1$ it reveals some hitherto unknown features of the harmonic oscillator eigenfunctions.

1. Introduction

Several years ago q-analogues of the harmonic oscillator were introduced to construct realizations of the quantum group $SU_q(2)$ (Macfarlane 1989, Biedenharn 1989, Sun and Fu 1989), which has appeared as the symmetry group of some exactly soluble models in two-dimensional statistical mechanics (Saleur 1989). This procedure generalizes a similar construction proposed by Schwinger for the $SU(2)$ group using ordinary harmonic oscillators. The q-oscillators have since become subject to intense investigation and have brought a host of new topics into research, for example q-classical mechanics (Shabanov 1992), q-deformed quantum mechanical potentials (Bonatsos *et al* 1991), q-supersymmetric systems (Spiridonov 1992), q-non-commutative phase space and q-Euclidean space (Wess and Zumino 1990) etc. It is not possible to review these developments here, we can only give an incomplete list of indicative references.

For a better understanding of the type of deformation we wish to propose for the Schrödinger picture of quantum mechanics with one degree of freedom, we review some recent works, which appear to be relevant to our proposal namely those dealing with realizations of either oscillator operators (i.e. creation and annihilation operators) or Heisenberg operators (i.e. position and momentum operators) in wavefunction spaces. Since we are not considering the classical limit of quantum mechanics we shall set $h/2\pi = 1$ to simplify the notation.

As originally proposed (Biedenharn 1989, Macfarlane 1989) the quantum mechanics of a q-oscillator is completely described by a q-oscillator algebra (Gilmore 1974) generated by the four operators $a^\dagger$, a, N and I obeying the commutators:

$$aa^\dagger - qa^\dagger a = q^{-N} \tag{1}$$

and

$$[N, a] = -a \qquad [N, a^\dagger] = a^\dagger. \tag{2}$$

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For $q = 1$ one recovers the usual oscillator algebra for which $a^\dagger$ (respectively a) is the creation (respectively annihilation) operator and N is the number operator. This deformation parallels the q-deformation of the $SU(2)$ algebra in which only the commutator between the raising and lowering operators has been changed. Mathematically such deformations may be understood in terms of Hopf algebras and Yang–Baxter algebras (Yan 1990) as well as quantum space analysis (Wess and Zumino 1990).

An equivalent form of this q-oscillator algebra may be obtained by the substitution $b = \sqrt{q}(q - q^{-1})aq^N$ and $b^\dagger = \sqrt{q}(q - q^{-1})q^N a^\dagger$ (Macfarlane 1989, Kulish and Damakinsky 1990):

$$bb^\dagger - q^2 b^\dagger b = I \tag{3}$$

and

$$[N, b] = -b \qquad [N, b^\dagger] = b^\dagger. \tag{4}$$

Curiously the commutator (3) had already been independently introduced by Kuryshkin as early as 1980 (Kuryshkin 1980, Janussis *et al* 1981). Although non-Fock representations of the q-oscillator algebra exist (Kulish 1991, Rideau 1992, 1993) it is the Fock representation which is most widely discussed (Biedenharn 1989) and its energy spectrum determined. There are essentially two classes of Fock representation in wavefunction spaces.

In the first class the b and $b^\dagger$ operators are represented by combinations of the Weyl operators of an auxiliairy variable x and its canonical conjugate momentum $-\mathrm{i}\,\mathrm{d}/\mathrm{d}x$ in the ordinary quantum mechanical sense. The Hamiltonian is $b^\dagger b$ up to a constant c-number and the operator N is essentially the logarithm of the Hamiltonian. However, there are two possible choices known so far for the explicit expressions of b and $b^\dagger$. The first choice made by Macfarlane (1989, Shabanov 1992) leads to a ground-state function which is the usual shifted Gaussian and excited states represented by the Rogers–Szegö polynomials in the space of this auxiliary degree of freedom x. The second choice which was made by Askey and Suslov (1993), yields energy eigenstate wavefunctions proportional to the Al-Salam Carlitz polynomials. In both cases the energy spectrum relative to the aforementioned Hamiltonian is not symmetric under q and q^{-1} exchange and not equal to the Biedenharn spectrum (Biedenharn 1989).

In the second class of Fock representation one deals with relation (1) and a is represented by a q-discrete derivative in a variable z, i.e.

$$D_q = \frac{q^{z(\mathrm{d}/\mathrm{d}z)} - q^{-z(\mathrm{d}/\mathrm{d}z)}}{qz - q^{-1}z} \tag{5}$$

whereas $a^\dagger$ is represented by the multiplication by z and N is simply $z(\mathrm{d}/\mathrm{d}z)$ (Kulish and Damakinsky 1990, Floratos 1991, Jurco 1991). We shall come back to this later in section 2.4. The energy eigenstate wavefunctions are simply monomials in z and the spectrum is identical to the Biedenharn spectrum. However, alternative choices for the q-discrete derivative exist but are generally not symmetric under q and q^{-1} exchange (Floreanini and Vinet 1991, 1993a).

$$D_q^{\pm} = \frac{1 - q^{\pm z(\mathrm{d}/\mathrm{d}z)}}{(1 - q^{\pm})z}. \tag{6}$$

A third possibility is to take for a and for $a^\dagger$ the sum and the difference between z and the q-discrete derivative in z; this was done by Li and Sheng (1992) who defined the q-oscillator main commutator to be

$$[a, a^\dagger] = \mu(N) \tag{7}$$

where $\mu(N)$ is an arbitrary function of N to be fixed by the chosen physics of the q-oscillator and N fulfils the commutation relations (2). The Hamiltonian is defined as $(a^\dagger a + 1/2\mu(N))$ and has a Fock spectrum which depends on the unknown function μ. In fact this realization of the q-oscillator algebra is quite close to the realization of a q-deformed Heisenberg algebra which is considered in this paper but differs on physical grounds leading to the deformation.

This brings us naturally to q-deformed Heisenberg algebra. For us a Heisenberg algebra is an algebra generated by three operators Q, P and I, which are, respectively, the position, conjugate momentum and identity operators. This algebra fixes the quantum kinematics and the dynamics is given by the choice of Hamiltonian operator. In the absence of deformation one has, as usual,

$$[Q, P] = \mathrm{i}I \tag{8}$$

and there exist two frameworks in which one may introduce a q-deformation of this commutator (8).

In the first setting, Q and P are elements of the so-called quantum line, the simplest case of quantum space. But then one must decide whether Q or P is Hermitian. If the momentum P is Hermitian $P = P^\dagger$ then there exists a Hermitian adjoint to Q, i.e. $Q^\dagger$. Thus we have the following relations:

$$\begin{aligned} &PQ - qQP = -\mathrm{i}I \\ &PQ^\dagger - q^{-1}Q^\dagger P = -\mathrm{i}q^{-1}I \\ &QQ^\dagger = qQ^\dagger Q. \end{aligned} \tag{9}$$

This choice was made by Schwenk and Wess (1992). But one may just as well take $Q = Q^\dagger$ and obtain

$$PQ - qQP = -\mathrm{i}I \qquad P^\dagger Q - q^{-1}QP^\dagger = -\mathrm{i}q^{-1}I. \tag{10}$$

This was considered by Ubriaco (1993), who also gave the following realization in wavefunction space:

$$Q \mapsto z \qquad P \mapsto -\mathrm{i}D_q^+ \qquad P^\dagger \mapsto -\mathrm{i}q^{-1}D_q^-. \tag{11}$$

The dynamics is obtained through the construction of a Hamiltonian which is Hermitian and consistent with the Hermicity of either Q or P. Extensions of these considerations to higher dimensional quantum spaces may be found in Carow-Watamura *et al* (1991), Carow-Watamura and Watamura (1993), Hebecker and Weich (1992) and Zumino (1991).

However, it is perhaps more natural to give up the previous quantum structure of the real line and directly consider a q-Heisenberg commutator. This was first pioneered by Minahan (1990) who introduced

$$PQ - q^2QP = -\mathrm{i}I \tag{12}$$

with the realization: $Q \mapsto z$ and $P \mapsto -\mathrm{i}D^{+}_{q^2}$. An alternative may be to start with q-classical mechanics and quantize to obtain a q-Heisenberg commutator: this was done by Arafeva and Volovich (19XX), who obtained in this way

$$PQ - qQP = -\mathrm{i}\sqrt{q}I. \tag{13}$$

They give analogous realizations: $Q \mapsto z$ and $P \mapsto -\mathrm{i}\sqrt{q}D^{+}_{q}$. Both authors have constructed Hermitian Hamiltonians and given a few low-lying eigenfunctions. Unfortunately if one interprets z as the coordinate of the q-oscillator, the corresponding wavefunctions have some unphysical features. Moreover, the eigenvalues are not symmetric under q and q^{-1} exchange and they are not of Biedenharn form.

Finally, as mentioned, before the work of Li and Sheng (1992) may be viewed alternatively as q-deformation of the Heisenberg algebra with the commutator:

$$[Q, P] = \mathrm{i}\mu(N_q) \tag{14}$$

where $\mu(z)$ is an undetermined function of z (Jannussis 1993). The Hermitian operator N_q will then be defined by

$$[N_q, Q] = -\mathrm{i}P \qquad [N_q, P] = \mathrm{i}Q. \tag{15}$$

The realization in wavefunction space for Q and P is simply $Q \mapsto z$ and $P \mapsto -\mathrm{i}\mathrm{D}_q$, which is manifestly symmetric under exchange of q and q^{-1}. Unfortunately there is no realization of N_q due partly to the unknown function $\mu(z)$.

To sum up the situation, we see that the so-called coordinate representation of the q-oscillator algebra under different forms does not seem to yield a bona fide Schrödinger picture for this q-oscillator. Moreover, several proposals for a q-Heisenberg algebra have realizations in function spaces which lead to unwanted aspects for the wavefunctions and the spectrum. In this unsatisfactory context, it is the objective of this paper to propose a q-deformation of the Heisenberg algebra for which a bona fide Schrödinger picture exists, the spectrum of the q-oscillator is the Biedenharn spectrum with a complete set of energy eigenfunctions explicitly caculated.

To this end we reconsider the quantum mechanics with one degree of freedom Q and its canonical conjugate momentum P. In the absence of deformation the quantum kinematics is given by $[Q, P] = \mathrm{i}I$. A Schrödinger picture exists via the realization $Q \mapsto z$ and $P \mapsto -\mathrm{i}z(\mathrm{d}/\mathrm{d}z)$ acting on square integrable wavefunctions of the coordinate z. Any q-deformation of this algebra, as we have seen in the work of Li and Sheng, would bring about a third operator and the corresponding commutators with Q and P. For Li and Sheng this operator is N_q, the analogue of the number operator for an oscillator. But, as we shall see, it seems more natural to us that this third operator should be rather the scaling generator M represented by $z(\mathrm{d}/\mathrm{d}z)$ in the wavefunction space. The commutators of M with Q and P are, in the absence of q-deformation,

$$[M, Q] = Q \qquad [M, P] = -P. \tag{16}$$

Now the deformation proposed here consists solely in the replacement of $\mathrm{d}/\mathrm{d}z$ by the q-discrete derivative as the representative of P:

$$\begin{aligned} &P \longmapsto \mathrm{i}^{-1} D_q \\ &D_q\psi(z) = \frac{\psi(qz) - \psi(q^{-1}z)}{qz - q^{-1}z}. \end{aligned} \tag{17}$$

This operator $\mathrm{i}^{-1}D_q$ exhibits an internal *q-symmetry* under the exchange $q \longleftrightarrow q^{-1}$, which, as we shall see, will play an important role in the determination of the wavefunction. This q-symmetry arises most naturally in the context of statistical mechanics of the six-vertex model where the coupling constant is expressed as $\frac{1}{2}(q+q^{-1}) = \Delta$, the so-called Lieb parameter (Pasquier and Saleur 1990, Kulish and Sklyanin 1991, Aizawa 1993). Appositely it turns out that this statistical model is at the origin of the concept of a quantum group, therefore it is plausible that q-symmetry would be expected in a physical system. One may interpret this new operator assignment for the canonical momentum as some *worsening* of the measurement of the momentum, i.e. the result of the measurement is *coarser* in a sense but it follows a definite prescription controlled by the parameter q which may be attributed in turn to the measuring apparatus.

Since we are keeping the same working framework as that normally used in quantum mechanics (i.e. $q = 1$) there exists a well-defined inner product in the Hilbert space of wavefunctions. This is in contrast to several authors who advocate a new inner product defined by a q-integral of the Jackson type (Li and Sheng 1992, Minahan 1990, Gray and Nelson 1990). In our opinion there is no clear quantum mechanical interpretation of this inner product at present and this is quite clear if one works in quantum spaces (Arafeva and Volovich 1991). We feel more confident in keeping the same old infrastructure and think of the q-deformation as *induced* by a *worsening* of the momentum measurement process. In this respect $-\mathrm{i}D_q$ as representative of P remains a Hermitian operator. Given two wavefunctions $\phi(z)$ and $\psi(z)$, the matrix element of P is

$$
\begin{aligned}
\langle\phi|P\psi\rangle &= \int \phi^*(z)(-\mathrm{i}D_q\psi(z))\,\mathrm{d}z \\
&= -\mathrm{i}\int \phi^*(z)\frac{\psi(qz)}{(qz-q^{-1}z)}\,\mathrm{d}z + \mathrm{i}\int \phi^*(z)\frac{\psi(q^{-1}z)}{(qz-q^{-1}z)}\,\mathrm{d}z.
\end{aligned}
$$

Now performing a change of variable in the last two integrals we arrive at

$$
\begin{aligned}
&= -\mathrm{i}\int \psi(z)\frac{\phi^*(q^{-1}z)}{(qz-q^{-1}z)}\,\mathrm{d}z + \mathrm{i}\int \psi(z)\frac{\phi^*(qz)}{(qz-q^{-1}z)}\,\mathrm{d}z \\
&= \int (P\phi)^*(z)\psi(z)\,\mathrm{d}z \\
&= \langle P\phi|\psi\rangle. \qquad (18)
\end{aligned}
$$

The other two operators Q and M are evidently Hermitians.

The paper is organized as follows. In section 2 we examine the consequences of the representation of the operator P by a discrete derivative: the ensuing deformation of the canonical commutation relation between P and Q as well as between the 'would be' creation and annihilation operators of a harmonic oscillator. We then observe the appearance of a new operator which may be called the q-deformed identity I_q and this turns out to be as important as the Hamiltonian operator. Next we establish that the $SU(1,1)$ dynamical symmetry of the harmonic oscillator is now taken over by the quantum group $SU_q(1,1)$ via the previous deformation (Kulish and Damakinsky 1990). We end section 2 with a comparision with the Bargmann approach (Kulish and Damakinsky 1990, Floratos 1991) from the point of view of differential operators in wavefunction space.

Section 3 is devoted to the construction of the q-Schrödinger picture. Our starting point is the requirement that the ground-state wavefunction should be annihilated by the 'would be' annihalation operator traditionally constructed out of the P and Q but for $q \neq 1$. This requirement seems to be generally adopted for two reasons: first it is true for $q = 1$, second the existence of such a vacuum is convenient in constructing Fock spaces for quantum field theory. This naturally leads to a q-Schrödinger equation for the eigenfunctions, moreover one may view it as a type of q-deformed eigenvalue problem. The complete set of eigenfunctions can be constructed by solving a recursion relation which appears to be a q-symmetric generalization of the recursion relation for the Meixner polynomials of the first kind (Chihara 1978). Curiously the three first excited states can be generated by repeated application of the 'would be' creation operator on the ground-state wavefunction whereas the higher excited states are generated in a different procedure. The eigenvalues of this problem turn out to be those found by Biedenharn (1989). Finally taking the limit $q \to 1$, we discover some new aspects of the harmonic oscillator wavefunctions and are led to some new version of the so-called q-hypergeometric functions that is invariant under the exchange of $q \leftrightarrow q^{-1}$. We conclude with a short summary of the highlights obtained as well as a list of unsolved problems and topics to be investigated in the future.

2. Algebraic aspects

2.1. *q-Heisenberg algebra*

The previous prescription leads to a modification of the Heisenberg algebra similar to the deformation of the $SU(2)$ group into a $SU_q(2)$ group. By looking at the action of the operators Q, P and M on wavefunctions $\psi(z)$ we can establish that

$$[Q, P] = \mathrm{i}I_q \qquad [M, Q] = Q \qquad [M, P] = -P. \tag{19}$$

Here the operator I_q replaces the identity I and is expressed as

$$I_q = \frac{\{M + \frac{1}{2}I\}_q}{\{\frac{1}{2}\}_q}. \tag{20}$$

In this equation and the following ones we use the shorthand notation

$$\{n\}_q = (q^n + q^{-n})/2 \tag{21}$$

$$[n]_q = (q^n - q^{-n})/(q - q^{-1}). \tag{22}$$

Incidently one can compute the anticommutator between Q and P:

$$\{Q, P\} = \frac{[M + \frac{1}{2}I]_q}{\mathrm{i}[\frac{1}{2}]_q}. \tag{23}$$

We observe that the *q-symmetry* is manifest in this deformation of the Heisenberg algebra. The realization of this q-Heisenberg algebra in wavefunction space has appeared without the q-symmetry in Dai *et al* (1991).

An immediate consequence is the effect on the Heisenberg *uncertainty relation* for position and momentum in a state described by a wavefunction $\psi(z)$:

$$(\Delta Q)(\Delta P) \geqslant \tfrac{1}{2}(\psi, I_q\psi) \geqslant \tfrac{1}{2}(\psi, \psi). \tag{24}$$

This is not surprising and is expected since right from the beginning we have more 'uncertainty' in the momentum measurement. Note that we only make use of conventional commutators and not q-commutators between Hermitian operators or observables, thus the occurrence of *simultaneous* but infinitely accurate measurements will not be possible and appears even *harder* than usual.

2.2. *Alternative form of the q-Heisenberg algebra*

We now introduce the complex linear transformations of Q and P which are normally used to define the creation and annihilation operators in the case of an ordinary oscillator:

$$\begin{aligned} \alpha_q &= (Q + \mathrm{i}P)/\sqrt{2} \\ \alpha_q^\dagger &= (Q - \mathrm{i}P)/\sqrt{2}. \end{aligned} \tag{25}$$

The q-Heisenberg algebra is now equivalently defined by the commutators:

$$[\alpha_q, \alpha_q^\dagger] = I_q \tag{26}$$

$$[M, \alpha_q] = \alpha_q^\dagger \qquad [M, \alpha_q^\dagger] = -\alpha_q. \tag{27}$$

Again we observe that only one commutator is modified by the q-deformation compared with the original commutators of the harmonic oscillator.

A new feature is that the anticommutator between α_q and $\alpha_q^\dagger$ may no longer be expressed uniquely in terms of N; it will be expressed instead in terms of P and Q:

$$\{\alpha_q^\dagger, \alpha_q\} = (P^2 + Q^2). \tag{28}$$

However, the right-hand side of this equation represents precisely the Hamiltonian operator H_q of a harmonic oscillator of unit frequency which one obtains by the *correspondence principle* in quantum mechanics. Combining the equations one may write equivalently:

$$\alpha_q \alpha_q^\dagger = \tfrac{1}{2}(H_q + I_q) \tag{29}$$

$$\alpha_q^\dagger \alpha_q = \tfrac{1}{2}(H_q - I_q). \tag{30}$$

These equations show that the two operators H_q and I_q are on the same footing if one considers α_q and $\alpha_q^\dagger$. This is illustrated by the following sets of commutators, where we have set $q = \exp\gamma$:

$$[\alpha_q^\dagger, I_q] = 2\sinh^2(\tfrac{1}{2}\gamma)\left(\frac{\cosh\gamma(M + \frac{1}{2}I)}{\cosh\frac{1}{2}\gamma}\alpha_q^\dagger - \frac{\sinh\gamma(M + \frac{1}{2}I)}{\sinh\frac{1}{2}\gamma}\alpha_q\right) \tag{31}$$

$$[\alpha_q, I_q] = 2\sinh^2(\tfrac{1}{2}\gamma)\left(\frac{\cosh\gamma(M + \frac{1}{2}I)}{\cosh\frac{1}{2}\gamma}\alpha_q - \frac{\sinh\gamma(M + \frac{1}{2}I)}{\sinh\frac{1}{2}\gamma}\alpha_q^\dagger\right). \tag{32}$$

As $q \to 1$, both right-hand sides go to zero as expected. Now for H_q we have

$$[H_q, \alpha_q^\dagger] = \cosh(\tfrac{1}{2}\gamma)\cosh(\gamma(M + \tfrac{1}{2}I))\alpha_q^\dagger - \sinh(\tfrac{1}{2}\gamma)\sinh(\gamma(M + \tfrac{1}{2}I))\alpha_q \tag{33}$$

$$[H_q, \alpha_q] = \sinh(\tfrac{1}{2}\gamma)\alpha_q^\dagger \sinh(\gamma(M + \tfrac{1}{2}I)) - \cosh(\tfrac{1}{2}\gamma)\alpha_q \cosh(\gamma(M + \tfrac{1}{2}I)). \tag{34}$$

For $q \to 1$ we recover the fact that $H_1 \simeq M$ (Kulish and Damakinsky 1990, Floratos 1991). However, H_q and I_q do not commute as will be shown in section 2.3.

2.3. Dynamical symmetry

It is known that the Hamiltonian operator of the ordinary harmonic oscillator may be considered as one of the generators of the $SU(1, 1)$ group if one constructs the other two generators by suitable bilinear combinations of the creation and annihilation operators. It is then said that the harmonic oscillator admits the $SU(1, 1)$ group as a *dynamical symmetry group*. Kulish and Damakinsky (1990) have shown that the q-oscillator of Macfarlane and Biedenharn admits, for the dynamical symmetry group, the group $SU_q(1, 1)$. Here we show that the situation is very similar. Let us introduce the operator $L_q = P^2 - Q^2$; together with M and H_q they build the following commutators:

$$[H_q, M] = 4L_q \qquad [L_q, M] = 4H_q \qquad [H_q, L_q] = \{1\}_q[2M+1]_q. \tag{35}$$

Defining now the linear combinations:

$$K_q^{\pm} = \frac{\mp H_q + L_q}{\sqrt{2}\cosh\gamma} \qquad K_q^z = \tfrac{1}{8}(2M+1) \tag{36}$$

we obtain the defining commutators of the quantum group $SU_q(1, 1)$:

$$[K_q^z, K_q^{\pm}] = \pm K_q^{\pm} \qquad [K_q^+, K_q^-] = -[2K_q^z]_{q^4}. \tag{37}$$

The corresponding Casimir operator is given by the usual formula:

$$C = [K_q^z]_{q^4}[K_q^z + I]_{q^4} - K_q^- K_q^+. \tag{38}$$

The first commutator $[H_q, M]$ is non-trivial, since I_q is an analytic function of M and it thus cannot commute with H_q.

2.4. Comments

Up to this point it seems useful to compare the q-deformation proposed above with the q-Bargmann representation of the Heisenberg algebra of a harmonic oscillator: the creation and annihilation operators are represented in the space of analytic functions by (Kulish and Damakinsky 1990, Floratos 1991):

$$\alpha_q \longmapsto D_q \qquad \alpha_q^{\dagger} \longmapsto z. \tag{39}$$

They now fulfil a q-commutator:

$$\alpha_q \alpha_q^{\dagger} - q^{\mp}\alpha_q^{\dagger}\alpha_q = q^{\pm N}. \tag{40}$$

This q-commutator is manifestly *not* q-symmetric and the operator N is *defined* by the two commutators:

$$[N, \alpha_q] = -\alpha_q \qquad [N, \alpha_q^{\dagger}] = \alpha_q^{\dagger} \tag{41}$$

which are needed in the Schwinger construction of the representation of the $SU_q(2)$ group with two of such q-oscillators.

As a consequence one may compute most of the interesting quantities in terms of the operator N:

$$\alpha_q \alpha_q^{\dagger} = [N+I]_q \qquad \alpha_q^{\dagger}\alpha_q = [N]_q. \tag{42}$$

Consequently we obtain the commutator:

$$[\alpha_q, \alpha_q^\dagger] = \frac{\{N + \frac{1}{2}I\}_q}{\{\frac{1}{2}\}_q}. \tag{43}$$

Finally the Hamiltonian operator $H_q = \alpha_q^\dagger \alpha_q + \alpha_q \alpha_q^\dagger$ may also be expressed in terms of N as

$$H_q = \frac{[N + \frac{1}{2}I]_q}{[\frac{1}{2}]_q}. \tag{44}$$

To make contact with the usual observables Q and P we compute their commutator using the linear combinations (9):

$$[Q, P] = \mathrm{i}\frac{\{N + \frac{1}{2}I\}_q}{\{\frac{1}{2}\}_q} = I_q \tag{45}$$

which is of the same form as the commutator of equation (7). The anticommutator between Q and P is, however, *not* expressible in terms of N:

$$\{Q, P\} = -\mathrm{i}(\alpha_q^\dagger)^2 - \mathrm{i}(\alpha_q)^2. \tag{46}$$

The commutator of Q and P leads, on the other hand, to the modified *uncertainty relation* in a state described by $\psi(z)$:

$$(\Delta P)(\Delta Q) \geqslant \tfrac{1}{2}(\psi, I_q \psi) \geqslant \tfrac{1}{2}(\psi, \psi) \tag{47}$$

which is identical to the previous one given by equation (8). However, in view of the linear combinations (9) and the assignments (23), the realizations of P and Q as differential operators are given by (compare with Li and Sheng (1992))

$$\begin{aligned} Q &\longmapsto (D_q + z)/\sqrt{2} \\ P &\longmapsto (D_q - z)/\sqrt{2}. \end{aligned} \tag{48}$$

The *uncertainty relation* (31) thus takes care of *both* Q and P built-in measurement uncertainties due to the presence of D_q. This is a crucial difference from our consideration which starts out with built-in uncertainties for the measurement of the momentum *only*. It turns out, as we shall see in the next section, that the structure of the energy eigenfunctions is then completely different, although the energy spectrum is the same. We thus recover the same relationship between the Schrödinger and Bargmann pictures before deformation ($q = 1$).

The Macfarlane–Biedenharn oscillator can be described advantageously as a representation space where N is diagonal. Such a function space is spanned by states that are obtained by repeated application of the q-creation operator on a vacuum state. Coherent states can then be defined easily as illustrated in many papers (Chaichian *et al* 1990, Katriel and Solomon 1991).

3. The q-Schrödinger picture

3.1. Ground-state wavefunction of the oscillator

Since the deformation introduced in this paper is due only to 'coarse' momentum measurements there is no reason to expect drastic changes in the wavefunctions. In particular, in the ground state we expect the wavefunction to retain its Gaussian form and we shall also assume, as almost everyone else does, that this ground-state wavefunction $\psi_0(z)$ is annihilated by the α_q operator (Macfarlane 1989, Minahan 1990):

$$\alpha_q \psi_0(z) = 0. \tag{49}$$

We recall that when $q \longrightarrow 1$, α_q tends to the usual annihilation operator. The previous equation is, in fact, a linear functional equation:

$$\psi_0(qz) - \psi_0(q^{-1}z) = -z^2(q - q^{-1})\psi_0(z). \tag{50}$$

The solution is, up to a normalization factor, a q-Gaussian function defined by the convergent series expansion

$$\psi_0(z) \simeq \sum_{p=0}^{\infty} \frac{(-1)^p z^{2p}}{[2]_q \ldots [2p]_q}. \tag{51}$$

By construction $\psi_0(z)$ is even under space parity and even under $q \longleftrightarrow q^{-1}$ exchange. This functional relation (35) also shows that for $z \geqslant 0$, $\psi_0(z)$ has no zero and decreases monotonically. Thus this is a physically acceptable wavefunction for the oscillator ground state, whereas the wavefunction proposed by Minahan (1990) does not have this property. Moreover, one may verify directly that $\psi_0(z)$ fulfils the following q-Schrödinger equation:

$$H_q \psi_0(z) = I_q \psi_0(z) \tag{52}$$

which may be explicitly written as

$$-D_q^2 \psi_0(z) + z^2 \psi_0(z) = \frac{q^{1/2}\psi_0(qz) + q^{-1/2}\psi_0(q^{-1}z)}{q^{1/2} + q^{-1/2}}. \tag{53}$$

This has the typical form of a q-deformed differential equation where ordinary derivatives are replaced by q-discrete derivatives and the 'right-hand side' of the equation contains the unknown function with scaled arguments (see, for example, Exton (1983)). Usually in mathematics, however, the discrete derivative is not q-symmetric and the 'right-hand side' of the equation contains only one scaled argument whereas here qz and $q^{-1}z$ appear simultaneously.

It is clear that at $q = 1$, $\psi_0(z)$ becomes the usual Gaussian function which satisfies the ordinary Schrödinger equation for a harmonic oscillator with unit frequency and eigenvalue $2\varepsilon_0 = 1$.

3.2. *The q-Schrödinger equation and low-lying eigenfunctions*

The analysis of the ground state leads us to suspect that perhaps the sought q-deformed Schrödinger equation for the eigenfunction $\psi_n(z)$ should have the form:

$$H_q\psi_n(z) = 2\varepsilon_n I_q\psi_n(z) \tag{54}$$

where $2\varepsilon_n$ is the corresponding energy eigenvalue. Since $\psi_0(z)$ is annihilated by the α_q operator it is natural to seek the eigenfunctions of excited states by repeated application of the $\alpha_q^\dagger$ on $\psi_0(z)$, a well known procedure for the usual harmonic oscillator. Surprisingly we find that

$$\psi_1(z) = (\alpha_q^\dagger)\psi_0(z) \qquad \psi_2(z) = (\alpha_q^\dagger)^2\psi_0(z) \qquad \psi_3(z) = (\alpha_q^\dagger)^3\psi_0(z) \tag{55}$$

are actually eigenfunctions of the proposed q-Schrödinger equation with the following energy eigenvalues:

$$2\varepsilon_j = \frac{[j+\frac{1}{2}]_q}{[\frac{1}{2}]_q} \qquad j = 1, 2, 3. \tag{56}$$

Incidently, this formula also gives the ground-state energy if we set $j = 0$. So it seems very natural that formula (40) should be the spectrum of our oscillator although $(\alpha_q^\dagger)^4\psi_0(z)$ is *not* an eigenfunction of the q-Schrödinger equation. Such a spectrum has already been found by Biedenharn for his q-oscillator and it is remarkable that one should find the same here. Our q-Schrödinger equation may be viewed as a *q-deformed eigenvalue problem* in the sense that we are looking for a set of functions which render H_q proportional *not* to I, the identity but rather to a *deformed* identity I_q. Thus these q-eigenfunctions, if they exist, will not be orthogonal in the usual sense but they will obey

$$(\psi_j, I_q\psi_i) \simeq \delta_{ij}. \tag{57}$$

In this respect we may attach to each eigenfunction in addition to its usual norm a q-norm, the meaning of which remains to be clarified. Up to now we have kept the probabilistic meaning of the usual scalar product and have not, as some authors have proposed, introduced a new q-integration scheme which, in our opinion, is hard to interpret in the context of the basic principles of quantum mechanics (Gray and Nelson 1990). Finally, since H_q and I_q do not commute with each other (see section 2.3), finding the solutions of the proposed q-deformed Schrödinger equation is a non-trivial task. In previous studies of the q-oscillator one has benefited from the special role of the N or (N_q) operator (Biedenharn 1989, Li and Sheng 1992, Askey and Suslov 1993) which usually commutes with the Hamiltonian but this is not the case here.

3.3. *Construction of the eigenfunctions*

We now show that a set of q-eigenfunctions can be explicitly constructed for the q-Schrödinger equation. It turns out that, curiously, the first three low-lying functions are identical to those given in the section above. The q-Schrödinger equation is invariant under space reflection ($z \longrightarrow -z$) as well as under ($q \longrightarrow q^{-1}$) exchange. Therefore an eigenfunction will be characterized by three quantum numbers:

$c = 1, 3$, labels even, odd eigenfunctions under *space parity*,

$a = +, -$, labels even, odd eigenfunctions under *q-parity*,

n is the energy quantum number for c and a given, it may itself be an even or odd integer.

Thus a solution of the q-Schrödinger equation will have the notation $\psi_n^c(z, a)$. The previously found low-lying eigenfunctions will now appear in the new notation as follows.

$$\begin{aligned} \psi_0(z) &= \psi_0^1(z, +) \qquad & \psi_1(z) &= \psi_0^3(z, +) \\ \psi_2(z) &= \psi_1^1(z, +) \qquad & \psi_3(z) &= \psi_1^3(z, +). \end{aligned} \tag{58}$$

Now the functional form of these eigenfunctions suggests that, for arbitrary eigenfunctions, one should look for a power series expression of the type (up to a normalization factor):

$$\psi_n^c(z, a) = \sum_{p=0}^{\infty} \frac{(-1)^p z^{2p+c^*}}{[2]_q \dots [2p]_q} S_p^c(n, a) \tag{59}$$

where $c = 1+2c^*$ and the coefficients $S_p^c(n, a)$ now fulfil the linear homogeneous three-way recursion relation:

$$[2p + c]_q S_{p+1}^c(n, a) - [2p]_q S_{p-1}^c(n, a) = 2\varepsilon_n^c \frac{\{2p + c/2\}_q}{\{\frac{1}{2}\}_q} S_p^c(n, a). \tag{60}$$

We already know some of these coefficients $S_p^1(0, +) = S_p^3(0, +) = 1$, $S_p^1(1, +) = [2p+\frac{1}{2}]_q$ and $S_p^3(1, +) = [2p + \frac{3}{2}]_q$.

Inspection of the recursion relation for large-p behaviour suggests the following ansatz for the coefficients:

$$S_p^c(n, a) = \sum_{j=0}^{\infty} (u_j^c q^{(2n-4j)p} - v_j^c q^{-(2n-4j)p}) \tag{61}$$

where the u_j^c and v_j^c are functions of n, a, c and q. Upon substitution of the ansatz in the recursion relation one may set the coefficients of an arbitrary term $q^{\pm(2n+2-4j)p}$ equal to zero for $j = 0, 1, \dots, \infty$.

For $j = 0$ we find that u_0^c and, by the Biedenharn formula, v_0^c are arbitrary and the general eigenvalue is precisely

$$2\varepsilon_n^c = \frac{[2n + c/2]_q}{[\frac{1}{2}]_q}. \tag{62}$$

For general $j > 0$ we obtain two-way recursion relations for the u_j^c and v_j^c:

$$\begin{aligned} u_j^c &= -u_{j-1}^c q^{-c} F_q(n, j, c) \\ v_j^c &= -v_{j-1}^c q^c F_q(n, j, c) \end{aligned} \tag{63}$$

where the term $F_q(n, j, c)$ is given by the q-symmetric expression:

$$F_q(n, j, c) = \left(\frac{[2n - 2j + 2]_q}{[2j]_q}\right) \left(\frac{\{2j - 2 + c/2\}_q}{\{2n - 2j + c/2\}_q}\right). \tag{64}$$

The solutions to these two way-recursion relations can be immediately written down as

$$
\begin{aligned}
u_j^c &= u_0^c(-q^{-jc})\phi_j^c(n) \\
v_j^c &= v_0^c(-q^{jc})\phi_j^c(n).
\end{aligned}
\tag{65}
$$

The $\phi_j^c(n)$ is a q-symmetric coefficient given by

$$
\phi_j^c(n) = \prod_{k=1}^{j} \left(\frac{[2n+2-2k]_q}{[2k]_q} \right) \left(\frac{\{2k-2+c/2\}_q}{\{2n-2k+c/2\}_q} \right) \tag{66}
$$

for $j = 1, \ldots, n$ whereas $\phi_0^c(n) = 1$ by construction and $\phi_j^c(n) = 0$ for $j = n+1, \ldots, \infty$. Inspection shows that we have a remarkable symmetry:

$$
\phi_j^c(n) = \phi_{n-j}^c(n) \tag{67}
$$

which shall be used to simplify the expressions of $S_p^c(n, a)$ appearing for the moment as

$$
S_p^c(n, a) = \sum_{j=0}^{n} (-1)^{jc} \phi_j^c(n) (u_0^c q^{(2n-4j)p-jc} - v_0^c q^{-(2n-4j)p+jc}). \tag{68}
$$

It is now clear that even q-parity $a = +$ may be realized by the condition $u_0^c + v_0^c = 0$ and odd q-parity by $u_0^c - v_0^c = 0$. We are now in a position to give the complete set of coefficients $S_p^c(n, a)$ which, because of the special symmetry of the $\phi_j^c(n)$, will depend on the even (odd) structure of $n = 2s(2s+1)$:

$$
\begin{aligned}
S_p^c(2s+1, \mp) &\simeq \pm(q^{(s+1/2)c} \pm q^{-(s+1/2)c}) S_p^c(2s+1) \\
S_p^c(2s, \mp) &\simeq \mp(q^{sc} \mp q^{-sc}) S_p^c(2s)
\end{aligned}
\tag{69}
$$

where we have defined two coefficients independently of the q-parity:

$$
\begin{aligned}
S_p^c(2s+1) &= \sum_{j=0}^{s} (-1)^{jc} \phi_j^c(2s+1) (q^{(2p+c/2)(2s+1-2j)} - q^{-(2p+c/2)(2s+1-2j)}) \\
S_p^c(2s) &= \sum_{j=0}^{s-1} (-1)^{jc} \phi_j^c(2s) (q^{(2p+c/2)(2s-2j)} + q^{-(2p+c/2)(2s-2j)}) + (-1)^{sc} \phi_s^c(2s).
\end{aligned}
\tag{70}
$$

Thus it appears that q-parity enters only as a trivial factor whereas space parity leads to energy level splitting. Discarding these factors we may consider only *reduced* wavefunctions of the form:

$$
\begin{aligned}
\psi_{2s+1}^c(z) &= \sum_{p=0}^{\infty} S_p^c(2s+1) \frac{(-1)^p z^{(2p+c^*)}}{[2]_q \ldots [2p]_q} \\
\psi_{2s}^c(z) &= \sum_{p=0}^{\infty} S_p^c(2s) \frac{(-1)^p z^{(2p+c^*)}}{[2]_q \ldots [2p]_q}.
\end{aligned}
\tag{71}
$$

It is understood that the normalization factors are omitted.

3.4. Properties of the eigenfunctions

The eigenfunctions $\psi_n^c(z)$ found above enjoy a remarkable property: they can be written as a finite weighted sum of scaled ground-state eigenfunctions $\psi_0^c(z)$. To see this we substitute the expressions for the coefficients $S_p^c(n)$ in the power series defining the $\psi_n^c(z)$ and reorder after exchanging the summation order. We get:

$$\psi_{2s+1}^c(z) = \sum_{j=0}^{s}(-1)^{jc}\phi_j^c(2s+1)\{q^{s-j+\frac{1}{2}}\psi_0^c(q^{2s+1-2j}z) - q^{-s+j-\frac{1}{2}}\psi_0^c(q^{-2s-1+2j}z)\} \qquad (72)$$

$$\psi_{2s}^c(z) = \sum_{j=0}^{s-1}(-1)^{jc}\phi_j^c(2s)\{q^{s-j}\psi_0^c(q^{2s-2j}z) + q^{-s+j}\psi_0^c(q^{-2s+2j}z)\}. \qquad (73)$$

One also encounters a similar structure for eigenfunctions of an oscillator in N-dimensional quantum space (Fiore 1992, Carow-Watamura and Watamura 1993), which can be alternatively described by a set of infinite number of creation operators acting on the ground-state eigenfunction $\psi_0^c(z)$:

$$\psi_n^c(z) = A^{\dagger}(n,c)\psi_0^c(z) \qquad (74)$$

where the creation operators $A^{\dagger}(n,c)$ are only functions of the scaling generator M:

$$A^{\dagger}(2s+1,c) = 2(q-q^{-1})\sum_{j=0}^{s}(-1)^{jc}\phi_j^c(2s+1)[(2s+1-2j)(M+\tfrac{1}{2}I)]_q \qquad (75)$$

$$A^{\dagger}(2s,c) = 2\sum_{j=0}^{s-1}(-1)^{jc}\phi_j^c(2s)\{(2s-2j)(M+\tfrac{1}{2}I)\}_q + (-1)^{sc}\phi_s^c(2s)I. \qquad (76)$$

These are energy-dependent creation operators: they do not have a universal form. Similar operators have been encountered by Fiore (1992) in his investigations of the n-dimensional harmonic oscillator in quantum space. Finally there seems to be no set of annihilation operators. The simplest non-trivial creation operators are, for $c = 1, 3$, given by

$$A^{\dagger}(1,c) \simeq [M+\tfrac{1}{2}I]_q. \qquad (77)$$

We may then verify that the action of $[M+\frac{1}{2}I]_q$ on the ground-state wavefunction is the *same* as the action of the creation operator $\alpha_q^{\dagger}$ on the same ground-state wavefunction $\psi_0^c(z)$: this is the reason why the low-lying eigenfunctions have been obtained by the usual procedure of 'creation', but as one goes to higher excited states there is a net deviation from this procedure.

Being a finite sum of n q-Gaussian functions each eigenfunction decreases at large values of z as a q-Gaussian function. In principle one should expect that, for given n, there exist exactly n zeros for $\psi_n^c(z)$. This is difficult to prove in general and we can convince ourselves with the verification that the low-lying eigenstates have the required number of zeros.

The first excited state $\psi_1(z) = \psi_0^3(z,+)$ by definition (see equation (43)) has a zero at the origin $z = 0$. The second excited state is given by the expression

$$\psi_2(z) = \psi_1^1(z,+) = \frac{(-2)}{(q^{1/2}+q^{-1/2})}(q^{1/2}\psi_0(qz) - q^{-1/2}\psi_0(q^{-1}z)). \qquad (78)$$

It has only one zero z_0 on the positive z-axis given by $q\psi_0(qz_0) = \psi_0(q^{-1}z_0)$. Similarly, the third excited state has an equivalent structure with a zero at the origin and another one z_0 on the positive z-axis as one can see from its expression:

$$\psi_3(z) = \psi_1^3(z,+) = 2z\left(\frac{q+q^{-1}}{q^{1/2}-q^{-1/2}}\right)(q^{3/2}\psi_0(qz) - q^{-3/2}\psi_0(q^{-1}z)). \tag{79}$$

3.5. The $q \to 1$ limit and some new aspects of the harmonic oscillator

In this limit the standard quantum mechanics of the harmonic oscillator is recovered. However, the search for eigenstates as power series of the form

$$\psi_n^c(z) = \sum_{p=0}^{\infty} \frac{(-1)^p z^{2p}}{(2p)!} s_p^c(n) \tag{80}$$

appears rather unsual since it does not factorize out the Gaussian part of the wavefunction. Now, since $S_p^c(n) \to s_p^c(n)$ in the limit $q \to 1$, we obtain instead of (44) the following recursion relation:

$$(p + \tfrac{1}{2}c)s_{p+1}^c - ps_{p-1}^c = \varepsilon s_p^c \tag{81}$$

which is, as shown by Meixner (1972), the recursion relation of a class of hypergeometric polynomials:

$$s_p^c = s_p(\tfrac{1}{2}(\varepsilon - \tfrac{1}{2}c), \tfrac{1}{2}c, 2) = {}_2F_1(-p, -\tfrac{1}{2}(\varepsilon - \tfrac{1}{2}c); \tfrac{1}{2}c; 2). \tag{82}$$

These polynomials have been studied under a different form by Gottlieb (Truong and Peschel 1990), and they are also called Meixner polynomials of the first kind by Chihara (1978). Meixner observed that the quantity $\frac{1}{2}(\varepsilon - \frac{1}{2}c)$ should be an integer n or, alternatively, ε should be chosen as

$$\varepsilon = \varepsilon_n^c = 2n + \tfrac{1}{2}c \tag{83}$$

which, with $c = 1, 3$, corresponds precisely to the usual quantized energy levels of the harmonic oscillator. Moreover, this condition leads to the symmetry property between p and n for the hypergeometric polynomials (Meixner 1972):

$$s_p(n) = s_n(p). \tag{84}$$

An immediate consequence of this result is the expansion of the Hermite polynomial $H_n(z)$ in terms of the Meixner polynomials $s_p(n, \frac{1}{2}c, 2)$:

$$H_n(z) = \mathcal{N}_n \exp\frac{z^2}{2} \sum_{p=0}^{\infty} s_p(n, \tfrac{1}{2}c, 2)\frac{(-1)^p z^{2p}}{(2p)!} \tag{85}$$

where $\mathcal{N}_n$ is a normalization factor. Comparision with a known formula

$$H_n(z) = 2^{n/2} \exp\tfrac{1}{2}z^2 D_n\left(z\sqrt{2}\right) \tag{86}$$

suggests that parabolic cylinder functions $D_n(z)$ may have power series representation in terms of Meixner polynomials of this type. Finally we would like to point out a curious coincidence: the Meixner polynomials have appeared in the diagonalization of a special class of corner transfer matrix in statistical mechanics which has a regularly spaced spectrum just as they have appeared here for the harmonic oscillator with an equidistant spectrum (Truong and Peschel 1990).

To give a practical example of what $s_p^c(n)$ might be, we list below some of them for $n = 0, 1, 2, 3$, where we set $\frac{1}{2}c = c'$ for simplicity:

$$\begin{aligned} s_p^c(0) &\simeq \text{constant} \\ s_p^c(1) &\simeq (2p + c') \\ s_p^c(2) &\simeq (p^2 + c'p + \tfrac{1}{4}c'(1 + c')) \\ s_p^c(3) &\simeq (p^3 + \tfrac{3}{2}c'p^2 + \tfrac{1}{4}(2 + 3c' + 3c'^2)p + \tfrac{1}{8}c'(2 + 2c' + c'^2)). \end{aligned} \tag{87}$$

Thus these $s_p^c(n)$ are, in fact, proportional to the hypergeometric polynomials ${}_2F_1(-p, -n; c'; -2)$. It is therefore of interest to compare them with the $S_p^c(n)$ before setting $q \to 1$. Again the four first are, using the notation $q = \exp\gamma$,

$$\begin{aligned} S_p^c(0) &\simeq C\text{constant} \\ S_p^c(1) &\simeq \sinh\gamma(2p + c') \\ S_p^c(2) &\simeq 2\cosh\gamma(4p + 2c') - \left(\frac{\sinh 4\gamma}{\sinh 2\gamma}\right)\left(\frac{\cosh\gamma c'}{\cosh(2 + c')\gamma}\right) \\ S_p^c(3) &\simeq \sinh\gamma(6p + 3c') - \sinh\gamma(2p + c')\left(\frac{\sinh 6\gamma}{\sinh 2\gamma}\right)\left(\frac{\cosh\gamma c'}{\cosh\gamma(4 + c')}\right). \end{aligned} \tag{88}$$

Clearly these are not polynomials in p but as $q \to 1$ or, equivalently, $\gamma \to 0$, their lowest non-vanishing order in γ terms are proportional to the previous $s_p^c(n)$. This fact leads us to conjecture that the $S_p^c(n)$ are obtainable from a class of q-symmetric hypergeometric functions of the type:

$${}_2\mathcal{F}_1(\{a\}, [b]; \{c\}; z) = \sum_{j=0}^{\infty} z^j \frac{(\{a\}_q)_j([b]_q)_j}{(\{c\}_q)_j([1]_q)_j} \tag{89}$$

where we have introduced the generalized Pochhammer symbols:

$$(\{a\}_q)_j = \{a\}_q\{a + 1\}_q \ldots \{a + j - 1\}_q \tag{90}$$

$$([b]_q)_j = [b]_q[b + 1]_q \ldots [b + j - 1]_q. \tag{91}$$

Then the $S_p^c(n)$ are linear combinations of these q-symmetric hypergeometric functions defined with q^2 instead of q and the parameter a now has the value $n = 2s + 1$ which terminates the hypergeometric series (89).

$$\begin{aligned} S_p^c(2s + 1) = {} & q^{(2p+c')}{}_2\mathcal{F}_1(\{c/4\}, [-n]; \{-n + 1 - (c/4)\}; q^{-(4p+c)}) \\ & - q^{-(2p+c')}{}_2\mathcal{F}_1(\{c/4\}, [-n]; \{-n + 1 - (c/4)\}; q^{(4p+c)}). \end{aligned} \tag{92}$$

A similar expression exists for $S_p^c(2s)$. The detailed study of these new q-functions is unfortunately beyond the scope of this paper and will be treated elsewhere. The important point is that the present treatment of the q-oscillator has led to the definition of new q-functions different from what one may find in the mathematical literature (Exton 1983, Floreanini and Vinet 1993b, Askey and Wilson 1984). Finally, for arbitrary real c the wavefunctions $\psi_n^c(z)$ of equation (55) may be regarded as a q-parabolic cylinder function.

4. Conclusions

Having observed that since the introduction of the q-commutator by Macfarlane and Biedenharn most treatments of the q-oscillator have been made in the occupational number formalism, we have tried to find an equivalent Schrödinger picture. Our starting point is the validity of standard quantum mechanics with its usual probabilistic interpretation. We explore only a specific deformation which consists in replacing the usual coordinate derivative by the q-symmetric discrete derivative as the representative of the momentum operator. In doing so we introduce built-in uncertainties into momentum measurements, which leads to a q-deformation. To construct a consistent Schrödinger representation we require, as most people do, that the ground-state wavefunction be annihilated by a 'formal' annihilation operator and then obtain the corresponding q-Schrödinger equation. The wavefunctions so obtained seems to have the global features of non-deformed wavefunctions and can be expressed in terms of new q-deformed Meixner polynomials of the first kind. Remarkably the energy spectrum is the same as in the Macfarlane–Biedenharn case. The limit of vanishing deformation yields the usual oscillator under novel aspects which suggest that the q-deformation is linked to some new version of q-hypergeometric functions.

It is clear that not every aspect of this q-Schrödinger picture has been completely treated. The physical meaning of the q-orthogonality remains to be established. Energy-independent creation and annihilation operators do not seem to exist in a straightforward manner. It may be that their structure hinges on the structure of the contiguous recursion relations of the new q-hypergeometric functions, and these have not been constructed yet. Two important problems have not been touched upon: the corresponding q-Heisenberg picture and the classical limit of the q-oscillator. In the literature one may find numerous works on these two subjects but it seems that none of them has considered these two problems in relation to a q-Schrödinger picture. We shall be concerned with these topics in the future, in particular the construction of coherent states viewed as semiclassical states and hope that once these issues are resolved many applications for the q-formalism will bring fruitful results in many-body problems (Floratos 1991), statistical physics (Chaichian *et al* 1993) and integrable systems (Bogoliubov and Bullough 1992).

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